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Initial correlations of the multiplicative process, driven by random jumps

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Abstract. We consider a vector-valued stochastic process, which is multiplicatively driven by the Markov jump process. We obtain a closed expression for the average of the vector process by solving the Burshtein equation for the marginal average. It is shown that the solution for $t > t_0$ requires knowledge of an initial correlation operator, due to the finite correlation time of the jump process. We derive the equation for the steady-state solution, which is applied to evaluate the stationary correlation functions. Then we discuss some specific limits of the jump process, which arise if we take the correlation time to be zero or infinite. For the special case $L(x) = A + xB$ we derive from the Burshtein equation a recurrence relation between the moments of the vector process. This relation is solved and it is shown how all initial moments at t_0 determine the moments for $t \geq t_0$. We apply the recurrence relation to solve the two-state and the three-state process more explicitly. It is pointed out that the occurring initial correlations cannot be neglected in general.

1. Introduction

The theory of multiplicative stochastic processes has a long history in physics, but the equations occurring can rarely be solved for the average. This is mainly due to the fact that the average of a product of stochastic quantities does not factorise in the product of the averages, although it can be argued that good approximations can be derived by assuming such factorisations (van Kampen 1976). This approach, which leads to Bourret's integral equation (Bourret 1962), can be justified if the driving stochastic process has a short correlation time. In the limit of a zero correlation time and a Gaussian Markov process (the phase-diffusion process) some averages factorise exactly and the solution for the average can be found (Fox 1972). An extension to non-Gaussian processes has recently been given by Arnoldus and Nienhuis (1983).

We consider a vector $\sigma(t)$ in some Hilbert space, which obeys the stochastic differential equation

$$\mathrm{i}\, \mathrm{d}\sigma/\mathrm{d}t = L(x(t))\sigma. \tag{1.1}$$

Here $L(x(t))$ is a linear operator on this space, and L will depend instantaneously on the real-valued stochastic process $x(t)$. The issue is always to solve (1.1) for the stochastic average $\langle\sigma(t)\rangle$. An initial value $\sigma(t_0)$ determines the stochastic solution of (1.1), which can formally be represented by the time-ordered exponential

$$\sigma(t) = \theta \exp\left(-\mathrm{i} \int_{t_0}^{t} L(x(s))\, \mathrm{d}s\right)\sigma(t_0) \tag{1.2}$$

with θ the Dyson time-ordering operator. After expansion of the exponential, we can replace the time-ordered integrals by multiple integrals, which yields the equivalent representation

$$\sigma(t)=\left(1+\sum_{n=1}^{\infty}(-\mathrm{i})^n\int_{t_0}^{t}\mathrm{d}t_n\int_{t_0}^{t_n}\mathrm{d}t_{n-1}\ldots\int_{t_0}^{t_2}\mathrm{d}t_1\,L(x(t_n))\ldots L(x(t_1))\right)\sigma(t_0). \qquad (1.3)$$

Commonly the initial state $\sigma(t_0)$ of the process is assumed to be non-stochastic and prescribed. Then we only have to average the time-ordered exponential. This has been done for the Ornstein–Uhlenbeck process (Dixit *et al* 1980, Zoller *et al* 1981, Yeh and Eberly 1981) for the N-state Markov jump process (Deng and Eberly 1984) and for the random-jump process (Brissaud and Frisch 1974, Shapiro and Loginov 1978). In many cases of practical interest $\sigma(t_0)$ is not known and depends on the past values of $x(t)$. We can only replace $\sigma(t_0)$ by its average $\langle\sigma(t_0)\rangle$ if the process $x(t)$ is delta-correlated, which would justify the factorisation. A stochastic initial value $\sigma(t_0)$ appears for instance in correlation functions of a quantum mechanical system, driven by an external stochastic process. A prime example would be the spectrum of the radiation, emitted by a two-state atom in a classical driving field (Arnoldus and Nienhuis 1986).

In this paper we take $x(t)$ to be the random-jump process, which has a finite correlation time. This time is not necessarily small, and hence a factorisation of averages would not be exact. Furthermore we allow the initial value $\sigma(t_0)$ to depend on the past in a stochastic way. We obtain an exact expression for the stochastic average $\langle\sigma(t)\rangle$ in a form that can be explicitly evaluated in specific cases. To our knowledge, a multiplicative process $\sigma(t)$, driven by a stochastic process with a finite correlation time, has never been solved without a factorisation assumption. Exact averages have only been obtained for delta-correlated processes, and for the opposite situation of an infinite correlation time. For this static case all averages reduce to trivial single-time averages (Kuś 1984, Arnoldus and Nienhuis 1985).

2. The random-jump process

The process $x(t)$ is taken to be a stationary homogeneous Markov process. Its stochastics is then fully determined by the probability distribution $P(x)$ and the conditional probability $P_\tau(x_2|x_1)$, which has the significance of the probability density for the occurrence of $x(t+\tau)=x_2$, when $x(t)=x_1$ (Stratonovich 1963). The time-independent probability distribution $P(x)$ must obviously obey the identity

$$\int \mathrm{d}x_1\,P_\tau(x_2|x_1)P(x_1)=P(x_2). \qquad (2.1)$$

The time evolution of P_τ is governed by the Master equation (van Kampen 1981)

$$(\partial/\partial\tau)P_\tau(x_3|x_1)=\int \mathrm{d}x_2(W(x_3,x_2)P_\tau(x_2|x_1)-W(x_2,x_3)P_\tau(x_3|x_1)) \qquad (2.2)$$

for $\tau\geqslant 0$, where $W(x',x)\geqslant 0$ has the significance of the transition rate from the value x to the value x'. The initial condition for (2.2) reads

$$P_0(x_2|x_1)=\delta(x_2-x_1). \qquad (2.3)$$

If we multiply (2.2) by $P(x_1)$, integrate over x_1 and apply relation (2.1) we obtain

$$\int \mathrm{d}x'\,W(x,x')P(x')=P(x)\int \mathrm{d}x'\,W(x',x). \qquad (2.4)$$

This imposes a constraint on the transition rate $W(x, x')$, which is necessary in order to preserve the stationarity of $x(t)$.

We now specialise the process by supposing that $x(t)$ can perform jumps at random instants, in such a way that the probability for a transition $x' \to x$ is independent of the initial value x'. This implies that $W(x, x')$ is independent of x', and with (2.4) we then find

$$W(x, x') = \gamma P(x) \tag{2.5}$$

with

$$\gamma = \int \mathrm{d}x \, W(x, x') > 0 \tag{2.6}$$

the jump rate, independent of the initial value x' before the jump. Equation (2.5) indicates that the probability distribution of $x(t)$ just after the jump is independent of the condition that a jump occurred. This picture is reminiscent of the strong-collision model for the velocity distribution of atoms in a gas (Rautian and Sobel'man 1967).

With relation (2.5) the Master equation attains the form

$$(\partial/\partial\tau) P_\tau(x_2|x_1) = -\gamma(P_\tau(x_2|x_1) - P(x_2)) \tag{2.7}$$

and with the initial condition (2.3) the solution is

$$P_\tau(x_2|x_1) = \mathrm{e}^{-\gamma\tau}\delta(x_2 - x_1) + (1 - \mathrm{e}^{-\gamma\tau})P(x_2) \qquad \tau \geq 0. \tag{2.8}$$

This reveals that the transition rate $\gamma > 0$ and the probability distribution $P(x)$ fix the stochastics of $x(t)$. Notice that we have made no restrictive assumptions about $P(x)$. By allowing $P(x)$ to contain a sum of δ functions, we would in fact have a discrete set of possible values of x. Hence our model, which is known as the Kubo-Anderson process (Kubo 1954, Anderson 1954), contains the random telegraph model as a special case (Wódkiewicz 1981).

From the Markov property and (2.8), we derive the relation for the moments of $x(t)$

$$\begin{aligned} \langle x(t_n)^k x(t_{n-1}) \ldots x(t_1)\rangle &= \exp[-\gamma(t_n - t_{n-1})]\langle x(t_{n-1})^{k+1} x(t_{n-2}) \ldots x(t_1)\rangle \\ &\quad + \{1 - \exp[-\gamma(t_n - t_{n-1})]\}\langle x^k\rangle\langle x(t_{n-1}) \ldots x(t_1)\rangle \end{aligned} \tag{2.9}$$

for $k = 0, 1, 2, \ldots$ and $t_n \geq \ldots \geq t_1$. Starting from $k = 1$, this result determines the moments $\langle x(t_n) \ldots x(t_1)\rangle$ recursively. Especially for $n = 2$ we obtain

$$\langle x(t_2)x(t_1)\rangle - \langle x\rangle^2 = \exp[-\gamma(t_2 - t_1)](\langle x^2\rangle - \langle x\rangle^2) \qquad t_2 \geq t_1 \tag{2.10}$$

expressed in the variance $\langle x^2\rangle - \langle x\rangle^2$. We see that we can identify γ^{-1} as the correlation time of the process, which equals the jump rate. It will turn out in § 6 that (2.9) is particularly useful to study the limits of long and small correlation times. From (2.9) we observe that $x(t)$ is not Gaussian in general, but in § 6 we will show that $x(t)$ has two Gaussian limits.

3. The Burshtein equation and its solution

The general stochastic solution of the multiplicative stochastic process $\sigma(t)$ is given by (1.3). If we assume a non-stochastic initial value $\sigma(t_0)$ and take the average term by term, then only the n-time average $\langle L(x(t_n)) \ldots L(x(t_1))\rangle$ for $t \geq t_n \geq \ldots \geq t_1 \geq t_0$

is required. With the Markov property of the n-fold probability distribution of $x(t)$ and the expression (2.8) for $P_\tau(x_2|x_1)$, we can formally write down the average. Then it appears that the resulting series can be summed again (Brissaud and Frisch 1971), which yields the average $\langle\sigma(t)\rangle$. This result however does not provide the general solution, since the initial value $\sigma(t_0)$ might depend stochastically on the past. Then the averages should be taken as $\langle L(x(t_n))\ldots L(x(t_1))\sigma(t_0)\rangle$. Due to the finite correlation time of the process $x(t)$, the average $\sigma(t_0)$ does not factorise from $\langle L(x(t_n))\ldots L(x(t_1))\rangle$.

An alternative approach to multiplicative processes starts from the Burshtein equation for the marginal average (Wódkiewicz *et al* 1984, Eberly *et al* 1984, Deng and Eberly 1984, Shore 1984). Introduce the vector

$$\zeta(x_0, t) = \langle\delta(x(t)-x_0)\sigma(t)\rangle. \tag{3.1}$$

It has the significance of $P(x_0)$ times the average of $\sigma(t)$ under the condition that $x(t)=x_0$, which is called the marginal average. We now multiply the expansion (1.3) by $\delta(x(t)-x_0)$ and formally take the average. Then the t dependence of $x(t)$ is transferred to a t dependence of the probability distribution as $P_{t-t_n}(x|x_n)$. If we differentiate the expansion term by term with respect to t, we can apply the Master equation (2.7) in the derivative of $P_{t-t_n}(x|x_n)$. If we sum the resulting series, we obtain the differential equation for the marginal average

$$\mathrm{i}\frac{\partial}{\partial t}\zeta(x_0, t) = L(x_0)\zeta(x_0, t) + \mathrm{i}\gamma\int (P(x_0) - \delta(x_0 - x_0'))\zeta(x_0', t)\,\mathrm{d}x_0' \tag{3.2}$$

which combines the Master equation and the evolution equation (1.1). This is the famous Burshtein equation (Burshtein 1965). Here $L(x_0)$ depends only parametrically on x_0, which is a great simplification in comparison with the stochastic equation (1.1), where $L(x(t))$ depends dynamically on the process $x(t)$.

If we can solve (3.2) for $\zeta(x_0, t)$, then $\langle\sigma(t)\rangle$ follows from

$$\langle\sigma(t)\rangle = \int \zeta(x_0, t)\,\mathrm{d}x_0 \tag{3.3}$$

as can be seen from (3.1). The solution of (3.2) will be hard to obtain in general, and depends on the choice of $P(x)$. With (3.3) however, we notice that it is sufficient to solve (3.2) for $\int \zeta(x_0, t)\,\mathrm{d}x_0$, rather than for $\zeta(x_0, t)$ itself. This is easily accomplished with a Laplace transform. If we define

$$\tilde{\zeta}(x_0, \omega) = \int_{t_0}^{\infty} \exp[\mathrm{i}\omega(t-t_0)]\zeta(x_0, t)\,\mathrm{d}t \tag{3.4}$$

then the Burshtein equation becomes

$$\tilde{\zeta}(x_0, \omega) = \frac{\mathrm{i}}{\omega + \mathrm{i}\gamma - L(x_0)}\zeta(x_0, t_0) + \frac{\mathrm{i}\gamma}{\omega + \mathrm{i}\gamma - L(x_0)}P(x_0)\int \tilde{\zeta}(x_0', \omega)\,\mathrm{d}x_0' \tag{3.5}$$

and integrating this equation over x_0 yields

$$[1 - \gamma G(\omega + \mathrm{i}\gamma)]\langle\tilde{\sigma}(\omega)\rangle = \int \mathrm{d}x_0 \frac{\mathrm{i}}{\omega + \mathrm{i}\gamma - L(x_0)}\zeta(x_0, t_0). \tag{3.6}$$

Here we have introduced the static resolvent

$$G(\omega) = \int \mathrm{d}x\, P(x)\frac{\mathrm{i}}{\omega - L(x)} \tag{3.7}$$

which is a single-time average. Now we can insert definition (3.1) for $\zeta(x_0, t_0)$ in (3.6) and perform the integration. We then obtain

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{1}{1-\gamma G(\omega+\mathrm{i}\gamma)}\left\langle\frac{\mathrm{i}}{\omega+\mathrm{i}\gamma-L(x(t_0))}\sigma(t_0)\right\rangle \tag{3.8}$$

for the Laplace transform of $\langle\sigma(t)\rangle$. This is the general solution for the average of the multiplicative process $\sigma(t)$. The factor in brackets in (3.8) contains the initial correlations at t_0. It is not a simple single-time average, because $\sigma(t_0)$ depends stochastically on its history. If $\sigma(t_0)$ would happen to be non-stochastic, then (3.8) would reduce to

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{1}{1-\gamma G(\omega+\mathrm{i}\gamma)} G(\omega+\mathrm{i}\gamma)\sigma(t_0) \tag{3.9}$$

which is the solution of Brissaud and Frisch (1974). Only in this case is $\langle\tilde{\sigma}(\omega)\rangle$, and therefore also $\langle\sigma(t)\rangle$, completely determined by the initial state $\sigma(t_0)$. In the appendix we cast the solution (3.8) in a different form, which will reveal more clearly the significance of the initial correlations.

4. Stationary state

The solution $\sigma(t)$ from (1.2) will exhibit persisting fluctuations, even in the limit $t\to\infty$, because $x(t)$ keeps on jumping. The average $\langle\sigma(t)\rangle$ however might reach a stationary state, as a result of an effective damping, caused by the fluctuations in $x(t)$. The general solution (3.8) takes the form of the integral equation

$$\langle\sigma(t)\rangle - \gamma\int_{t_0}^{t}\mathrm{d}t'\exp[-\gamma(t-t')]\int\mathrm{d}x\,P(x)\exp[-\mathrm{i}L(x)(t-t')]\langle\sigma(t')\rangle$$
$$= \exp[-\gamma(t-t_0)]\langle\exp[-\mathrm{i}L(x(t_0))(t-t_0)]\sigma(t_0)\rangle \tag{4.1}$$

in the time domain. In the limit $t\gg t_0$ the right-hand side of (4.1) vanishes, because of the first exponential. Hence in this long-time limit, the dependence on the initial correlations disappears.

Let us now assume that $\langle\sigma(t)\rangle$ reaches a steady state, which will be denoted by

$$\bar{\sigma} = \lim_{t\to\infty}\langle\sigma(t)\rangle. \tag{4.2}$$

This $\bar{\sigma}$ is not necessarily unique. If $\langle\sigma(t)\rangle$ attains a stationary value in the limit $t\gg t_0$, we can take $\langle\sigma(t')\rangle$ in (4.1) outside the integral as $\bar{\sigma}$, because the factor $\exp[-\gamma(t-t')]$ makes the contribution from $t'\ll t$ negligible. Then we can perform the t' integration, which yields the equation for the steady state

$$[1-\gamma G(\mathrm{i}\gamma)]\bar{\sigma} = 0. \tag{4.3}$$

If the operator $1-\gamma G(\mathrm{i}\gamma)$ has more than one eigenvalue zero, the steady state is not unique, and if it has no eigenvalue zero, the steady state is $\bar{\sigma}=0$. In this derivation we obviously supposed that the static evolution operator $\exp[-\mathrm{i}L(x)(t-t_0)]$ has no exponentially increasing components.

In the Laplace domain the steady state follows from

$$\bar{\sigma} = \lim_{\omega\to\mathrm{i}0^+} -\mathrm{i}\omega\langle\tilde{\sigma}(\omega)\rangle. \tag{4.4}$$

Then we can multiply (3.8) by $-\mathrm{i}\omega$ and take the limit $\omega\to \mathrm{i}0^+$. We notice that

$$\lim_{\omega\to \mathrm{i}0^+} -\mathrm{i}\omega \frac{\mathrm{i}}{\omega+\mathrm{i}\gamma-L(x(t_0))}=0 \tag{4.5}$$

and that $G(\omega+\mathrm{i}\gamma)$ exists for $\omega\to 0$. This gives again (4.3).

In this fashion it is easy to obtain the stationary solution of the Burshtein equation. If we introduce the steady-state marginal average as

$$\bar{\zeta}(x_0)=\lim_{t\to\infty}\zeta(x_0, t) \tag{4.6}$$

then it follows directly from (3.5) that $\bar{\zeta}(x_0)$ is related to $\bar{\sigma}$ as

$$\bar{\zeta}(x_0)=P(x_0)\frac{\mathrm{i}\gamma}{\mathrm{i}\gamma-L(x_0)}\bar{\sigma}. \tag{4.7}$$

After solving (4.3) for $\bar{\sigma}$, this equation determines $\bar{\zeta}(x_0)$. Conversely, if we integrate (4.7) over x_0, we again find (4.3).

5. Correlation functions

Suppose that the system is prepared at t_0 in a non-stochastic state $\sigma(t_0)$. Then the solution for $\langle\sigma(t)\rangle$ with $t\geqslant t_0$ is given by the Laplace inverse of (3.9) and the initial correlations do not enter. On the other hand, in the case that (1.1) is a regression equation for a correlation function, the initial value $\sigma(t_0)$ corresponds to the equal-time correlation. Consider for example the quantum correlation $\langle F(t_0)G(t)\rangle$ between the two Heisenberg operators F and G. Transformation of this expression to the Schrödinger picture yields

$$\langle F(t_0)G(t)\rangle=\mathrm{Tr}\, GU(t, t_0)(\rho(t_0)F) \tag{5.1}$$

with $U(t, t_0)$ the evolution operator in Liouville space for the density matrix $\rho(t)$. This can be written as $\mathrm{Tr}\, G\sigma(t)$ if we define $\sigma(t)$ as

$$\sigma(t)=U(t, t_0)(\rho(t_0)F). \tag{5.2}$$

Hence the time evolution of $\sigma(t)$ also obeys the Liouville equation, which is assumed to be of the multiplicative form (1.1). This turns the two-time correlation (5.1) into a stochastic quantity.

The initial value $\sigma(t_0)$, which determines the equal-time correlation, can be written as

$$\sigma(t_0)=\rho(t_0)F=R\rho(t_0) \tag{5.3}$$

where the action of the Liouville operator R is defined as the multiplication with F. This $\rho(t_0)$ is the density matrix of the quantum system at t_0, and hence its evolution from the initial state $\rho(0)$ with $0<t_0$ is also governed by (1.1). We will only consider stationary correlation functions, which requires $t_0\gg 0$. Then, even if the initial state $\rho(0)$ is non-stochastic, the initial value $\sigma(t_0)=R\rho(t_0)$ for the $\sigma(t)$ process will stochastically depend on the evolution in $[0, t_0]$.

Although $\sigma(t)$ and $\rho(t)$ often obey the same equation, it is sometimes necessary to allow the equation for $\rho(t)$ to have a slightly different form (Zoller and Ehlotzky 1977, Arnoldus and Nienhuis 1983). Therefore we assume that $\rho(t)$ is determined by

$$\mathrm{i}\frac{\mathrm{d}\rho}{\mathrm{d}t} = L'(x(t))\rho \qquad t \leqslant t_0 \tag{5.4}$$

where L' is not necessarily equal to L. However, the driving process $x(t)$ is the same as in (1.1) for the evolution of $\sigma(t)$ for $t \geqslant t_0$.

We suppose that at time t_0, $\rho(t_0)$ has been driven by (5.4) long enough, so that $\langle\rho(t_0)\rangle$ has reached its steady-state $\bar{\rho}$, as discussed in the previous section. This $\bar{\rho}$ is the solution of

$$\left(1 - \int \mathrm{d}x\, P(x)\frac{\mathrm{i}\gamma}{\mathrm{i}\gamma - L'(x)}\right)\bar{\rho} = 0 \tag{5.5}$$

and the corresponding marginal average can be found from (4.7) with the substitutions $L \to L'$ and $\bar{\sigma} \to \bar{\rho}$. For the evaluation of the correlation function $\langle\tilde{\sigma}(\omega)\rangle$, we return to (3.6). The initial marginal average $\zeta(x_0, t_0)$ of $\sigma(t_0)$ can be written as

$$\zeta(x_0, t_0) = R\langle\delta(x(t_0) - x_0)\rho(t_0)\rangle \tag{5.6}$$

because of (5.3). In this expression we cannot replace $\rho(t_0)$ by $\bar{\rho}$, but the marginal average $\langle\delta(x(t_0) - x_0)\rho(t_0)\rangle$ has reached a steady state, which is related to $\bar{\rho}$ by (4.7) with $L \to L'$ and $\bar{\sigma} \to \bar{\rho}$. Hence we find

$$\zeta(x_0, t_0) = RP(x_0)\frac{\mathrm{i}\gamma}{\mathrm{i}\gamma - L'(x_0)}\bar{\rho} \tag{5.7}$$

in the stationary state. If we substitute (5.7) into (3.6) we obtain

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{1}{1 - \gamma G(\omega + \mathrm{i}\gamma)}\left(\int \mathrm{d}x\, P(x)\frac{\mathrm{i}}{\omega + \mathrm{i}\gamma - L(x)}R\frac{\mathrm{i}\gamma}{\mathrm{i}\gamma - L'(x)}\right)\bar{\rho}. \tag{5.8}$$

This result determines explicitly the stochastic average of a correlation function, including the initial correlations. Since this expression cannot be written as an evolution operator acting on the average initial state $R\bar{\rho}$, a factorisation assumption cannot be exact. Only for small correlation times γ^{-1}, in comparison with the eigenvalues of $L'(x)^{-1}$, we have

$$\lim_{\gamma\to\infty}\frac{\mathrm{i}\gamma}{\mathrm{i}\gamma - L'(x)} = 1 \tag{5.9}$$

which turns equation (5.8) into

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{1}{1 - \gamma G(\omega + \mathrm{i}\gamma)}G(\omega + \mathrm{i}\gamma)R\bar{\rho} \tag{5.10}$$

with $R\bar{\rho} = \langle\sigma(t_0)\rangle$. Comparison with (3.9) shows that only in the limit $\gamma^{-1} \to 0$, can this factorisation be justified.

6. Limits of small and long correlation time

The magnitude of the correlation time γ^{-1} has a great impact on the structure of the solutions. Let us first consider the limit of very frequent jumps, so $\gamma^{-1} \to 0$. Then the conditional probability (2.8) reduces to

$$P_\tau(x_2|x_1) = P(x_2) \tag{6.1}$$

for every $\tau \geqslant 0$, and hence this conditional probability becomes independent of the condition. Equation (6.1) holds generally only for $\gamma\tau \to \infty$, which illustrates that for long delays τ, the memory of the process to $x(t_1) = x_1$ vanishes. Now (6.1) holds for every $\tau \geqslant 0$, which implies that the process has no memory. If we take $\gamma^{-1} \to 0$ in the recurrence relation (2.9) for the moments of $x(t)$ we find

$$\langle x(t_n) \ldots x(t_1)\rangle = \langle x\rangle^n. \tag{6.2}$$

This reveals clearly that a high jump rate γ effectively turns $x(t)$ into $\langle x\rangle$. Therefore, also the multiplicative process $\sigma(t)$ will only respond to $\langle x\rangle$. This can also be found explicitly from the expansion (1.3). If we apply (6.1) and average the series (1.3), we can evaluate the multiple integrals. If we differentiate the result, we obtain

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)\langle\sigma(t)\rangle = \bar{L}\langle\sigma(t)\rangle \tag{6.3}$$

with

$$\bar{L} = \int L(x)P(x)\,\mathrm{d}x \tag{6.4}$$

the average of $L(x)$. The solution of (6.3) for $t \geqslant t_0$ is determined by $\langle\sigma(t_0)\rangle$, which proves in general the factorisation in the limit $\gamma^{-1} \to 0$. In the appendix we digress a little more on this limit.

Let us now consider the opposite case of an infinite correlation time γ^{-1}. With (2.5) we then find

$$W(x, x') = 0. \tag{6.5}$$

In this limit $\gamma \to 0$ the transition rate vanishes, which means that for every realisation of the process $x(t)$, the value of $x(t)$ is constant. This is the static limit. The conditional probability (2.8) becomes

$$P_\tau(x_2|x_1) = \delta(x_2 - x_1) \tag{6.6}$$

for every τ. This shows that $\sigma(t)$ depends only parametrically on x, and that averaging reduces to single-time averaging with $P(x)$. The moments of $x(t)$ become

$$\langle x(t_n) \ldots x(t_1)\rangle = \langle x^n\rangle \tag{6.7}$$

and the average of $\sigma(t)$ can be written as

$$\langle\sigma(t)\rangle = \int \mathrm{d}x\, P(x)\sigma(x, t) \tag{6.8}$$

where the non-stochastic $\sigma(x, t)$ is the solution of

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)\sigma(x, t) = L(x)\sigma(x, t). \tag{6.9}$$

The initial state $\sigma(x, t_0)$ depends also on x, which implies that a factorisation in this limit $\gamma^{-1} \to \infty$ is certainly not correct.

The probability distribution $P(x)$ is still arbitrary. If we take $P(x)$ as a Gaussian in the limit $\gamma \to 0$, then the process $x(t)$ is also Gaussian. The only stationary Gaussian Markov process is the Ornstein–Uhlenbeck process (Wax 1954), so this limit is the static limit of the Ornstein–Uhlenbeck process (Kuś 1984, Swain 1984).

Another interesting limit arises if we take again $\gamma^{-1} \to 0$, but allow the fluctuations in $x(t)$ to become very large, in such a way that the parameter

$$\lambda = \langle x^2 \rangle / \gamma \tag{6.10}$$

remains finite. Furthermore we take $P(x)$ symmetric, e.g. $P(-x) = P(x)$, which implies $\langle x \rangle = 0$. With the limit

$$\lim_{\gamma \to \infty} \gamma \exp[-\gamma(t_2 - t_1)] = \delta(t_2 - t_1 - 0^+) = 2\delta(t_2 - t_1) \qquad t_2 \geqslant t_1 \tag{6.11}$$

the moments of $x(t)$ become

$$\langle x(t_n) \ldots x(t_1) \rangle = (2\lambda)^n \delta(t_n - t_{n-1}) \ldots \delta(t_4 - t_3)\delta(t_2 - t_1) \tag{6.12}$$

for n even and $t_n \geqslant \ldots \geqslant t_1$, whereas the odd moments vanish.

This process $x(t)$ is also Gaussian and identical to the phase-diffusion process or Gaussian white noise (Fox 1972). It might seem that the moments (6.12) do not have the Gaussian property, but in the summation over the different permutations of time arguments on the right-hand side of (6.12), as it appears in the general form for the moments of a Gaussian process, only one term contributes for a specific time ordering, due to the exact delta correlations. The process $x(t)$ is also a Markov process. It is the Ornstein–Uhlenbeck process with zero correlation time and infinite variance.

7. The moment expansion

An important specific case arises if we take $L(x(t))$ as the linear form

$$L(x(t)) = A + x(t)B \tag{7.1}$$

with A and B non-commuting operators. This equation describes, for instance, the state of an atom in a multimode laser field, the fluorescence spectrum and the photon correlations. In this section we will elaborate the notion of initial correlations, and take advantage of the specific form of $L(x(t))$.

We introduce the moments of $\sigma(t)$ as

$$\Pi_k(t) = \langle x(t)^k \sigma(t) \rangle \qquad k = 0, 1, 2, \ldots \tag{7.2}$$

and in particular we have

$$\Pi_0(t) = \langle \sigma(t) \rangle. \tag{7.3}$$

The moments $\Pi_k(t)$ are determined by the marginal average $\zeta(x_0, t)$ according to

$$\Pi_k(t) = \int x_0^k \zeta(x_0, t)\, \mathrm{d}x_0 \tag{7.4}$$

as can be seen from (3.1). For $k = 0$ this reduces to (3.3). If we multiply the Burshtein equation (3.2) with x_0^k, insert (7.1) for $L(x(t))$ and integrate over x_0, we find an equation for the moments. We obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\Pi_k(t) = (A - \mathrm{i}\gamma)\Pi_k(t) + B\Pi_{k+1}(t) + \mathrm{i}\gamma\langle x^k \rangle \Pi_0(t) \tag{7.5}$$

which is a recurrence relation between the moments with different k values. This expression relates $\Pi_k(t)$ to $\Pi_{k+1}(t)$, $\Pi_0(t)$ and the static moments $\langle x^k \rangle$ of $P(x)$.

Equation (7.5) for the moments of $\sigma(t)$ is an infinite set of coupled first-order differential equations, which has to be solved with a given set of initial moments $\Pi_k(t_0)$. The solution of (7.5) relates the set of moments $\Pi_k(t)$ for $t \geqslant t_0$ to the initial set $\Pi_k(t_0)$. This shows that especially the state of the system $\Pi_0(t) = \langle\sigma(t)\rangle$ for $t \geqslant t_0$ will depend on all initial moments $\Pi_k(t_0)$, which constitute the initial correlations of $\sigma(t_0)$.

With the differential equation (1.1) and the specific form (7.1) for L, we can cast (7.5) in the form

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle x(t)^k\sigma(t)\rangle - \left\langle x(t)^k \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\sigma(t)\right\rangle = \mathrm{i}\gamma(\langle x(t)^k\rangle\langle\sigma(t)\rangle - \langle x(t)^k\sigma(t)\rangle) \tag{7.6}$$

which is the differentiation formula of Shapiro and Loginov (1978).

The coupled differential equations (7.5) can be transformed to algebraic equations by a Laplace transform. If we define

$$\tilde{\Pi}_k(\omega) = \int_{t_0}^{\infty} \exp[\mathrm{i}\omega(t-t_0)]\Pi_k(t)\,\mathrm{d}t \tag{7.7}$$

then (7.5) is equivalent to

$$(\omega - A + \mathrm{i}\gamma)\tilde{\Pi}_k(\omega) = \mathrm{i}\Pi_k(t_0) + B\tilde{\Pi}_{k+1}(\omega) + \mathrm{i}\gamma\langle x^k\rangle\tilde{\Pi}_0(\omega). \tag{7.8}$$

We can solve (7.8) recursively in terms of the moments of the static resolvent

$$G_k(\omega) = \left\langle x^k \frac{\mathrm{i}}{\omega - A - xB}\right\rangle = \int \mathrm{d}x\, P(x)x^k \frac{\mathrm{i}}{\omega - A - xB}. \tag{7.9}$$

This yields

$$\tilde{\Pi}_k(\omega) = \gamma G_k(\omega + \mathrm{i}\gamma)\tilde{\Pi}_0(\omega) + \frac{\mathrm{i}}{\omega - A + \mathrm{i}\gamma}\sum_{j=0}^{\infty}\left(B\frac{1}{\omega - A + \mathrm{i}\gamma}\right)^j \Pi_{k+j}(t_0) \tag{7.10}$$

in terms of the operators $G_k(\omega + \mathrm{i}\gamma)$ and the initial moments $\Pi_k(t_0)$. Equation (7.10) expresses $\tilde{\Pi}_k(\omega)$ in $\tilde{\Pi}_0(\omega)$, and if we take $k = 0$ in (7.10), we find for $\tilde{\Pi}_0(\omega)$ the explicit result

$$\tilde{\Pi}_0(\omega) = \frac{1}{1 - \gamma G_0(\omega + \mathrm{i}\gamma)}\frac{\mathrm{i}}{\omega - A + \mathrm{i}\gamma}\sum_{j=0}^{\infty}\left(B\frac{1}{\omega - A + \mathrm{i}\gamma}\right)^j \Pi_j(t_0) \tag{7.11}$$

which is the Laplace transform of $\langle\sigma(t)\rangle$. We notice that this solution involves all initial moments $\Pi_j(t_0)$. This shows again that for the random-jump process the average $\langle\sigma(t)\rangle$ is not determined by $\langle\sigma(t_0)\rangle$ alone in general, as would be the case if an initial factorisation was justified.

If we recall that the initial correlations are defined as $\Pi_k(t_0) = \langle x(t_0)^k\sigma(t_0)\rangle$, then we can write the solution (7.11) in a more condensed form. We obtain

$$\tilde{\Pi}_0(\omega) = \frac{1}{1 - \gamma G_0(\omega + \mathrm{i}\gamma)}\left\langle\frac{\mathrm{i}}{\omega - A + \mathrm{i}\gamma - x(t_0)B}\sigma(t_0)\right\rangle \tag{7.12}$$

which recovers the result (3.8). Similarly (7.10) reduces to

$$\tilde{\Pi}_k(\omega) = \gamma G_k(\omega + \mathrm{i}\gamma)\tilde{\Pi}_0(\omega) + \left\langle x(t_0)^k\frac{\mathrm{i}}{\omega - A + \mathrm{i}\gamma - x(t_0)B}\sigma(t_0)\right\rangle. \tag{7.13}$$

If we assume $P(x)$ to be symmetric, we can obtain an alternative expression for $G_k(\omega)$. With some effort we then find for k even

$$G_k(\omega)=\left\langle x^k \frac{\mathrm{i}}{\omega-A+\mathrm{i}x^2B[\mathrm{i}/(\omega-A)]B}\right\rangle \qquad k=0,2,4,\ldots \tag{7.14}$$

and the resolvents for k odd then follow from

$$G_k(\omega)=\frac{1}{\omega-A}BG_{k+1}(\omega) \qquad k=1,3,5,\ldots. \tag{7.15}$$

It will turn out that these relations are especially useful for the random telegraph process.

If the initial state $\sigma(t_0)$ is non-stochastic, then the initial moments reduce to

$$\Pi_k(t_0)=\langle x^k\rangle\sigma(t_0). \tag{7.16}$$

After substitution in the series expansion (7.11), the series can be resummed and we recover the result (3.9). In general however, the initial state will be stochastic, as for instance for a correlation function. The steady-state correlation function now simply follows from $t_0\to\infty$ in the initial moments. The long-time solution

$$\bar{\Pi}_k=\lim_{t_0\to\infty}\Pi_k(t_0)=\lim_{\omega\to\mathrm{i}0^+}-\mathrm{i}\omega\tilde{\Pi}_k(\omega) \tag{7.17}$$

is directly found from (7.10). We have

$$\bar{\Pi}_k=\gamma G_k(\mathrm{i}\gamma)\bar{\Pi}_0 \tag{7.18}$$

which expresses $\bar{\Pi}_k$ in $\bar{\Pi}_0$, and reduces to (4.3) for $k=0$. Substitution of $\bar{\Pi}_j$ for $\Pi_j(t_0)$ in (7.11) then yields the stationary correlation function.

8. The two-state process

The most elementary random-jump process is the random telegraph signal. In this special case the process $x(t)$ can only assume the values $\pm a$, with equal probability $P(\pm a)=\frac{1}{2}$. The simplifications are due to the property

$$x(t)^2=a^2 \tag{8.1}$$

for every t, which implies for the moments (7.2)

$$\Pi_{2k}(t)=a^{2k}\Pi_0(t) \qquad \Pi_{2k+1}(t)=a^{2k}\Pi_1(t) \qquad k=0,1,2,\ldots. \tag{8.2}$$

Hence all moments are determined by the initial correlations $\Pi_0(t_0)$ and $\Pi_1(t_0)$. We can substitute (8.2) in the general solution (7.11) and sum the two remaining series, but the result is rather cumbersome. It is more convenient to go back to the set of equations (7.5) for the moments. Because of (8.2) this infinite set truncates after the second equation, and we are left with only two remaining equations:

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)\Pi_0(t)=A\Pi_0(t)+B\Pi_1(t) \tag{8.3}$$

$$\mathrm{i}(\mathrm{d}/\mathrm{d}t)\Pi_1(t)=(A-\mathrm{i}\gamma)\Pi_1(t)+a^2B\Pi_0(t). \tag{8.4}$$

This set is easily solved with a Laplace transform, with the result

$$\tilde{\Pi}_0(\omega)=\frac{\mathrm{i}}{\omega-A+\mathrm{i}a^2B[\mathrm{i}/(\omega-A+\mathrm{i}\gamma)]B}\left(\Pi_0(t_0)-\mathrm{i}B\frac{\mathrm{i}}{\omega-A+\mathrm{i}\gamma}\Pi_1(t_0)\right) \tag{8.5}$$

$$\tilde{\Pi}_1(\omega)=\frac{\mathrm{i}}{\omega-A+\mathrm{i}\gamma+\mathrm{i}a^2B[\mathrm{i}/(\omega-A)]B}\left(\Pi_1(t_0)-\mathrm{i}a^2B\frac{\mathrm{i}}{\omega-A}\Pi_0(t_0)\right). \tag{8.6}$$

This general solution is expressed in the two initial correlations $\Pi_0(t_0)$ and $\Pi_1(t_0)$. In a factorisation approximation the second term in large brackets in (8.5) is neglected (Wódkiewicz *et al* 1984). We wish to emphasise however that this term is not necessarily small.

The equation for the steady-state $\bar{\Pi}_0$ follows immediately from (8.5). We find

$$\left(A+\mathrm{i}a^2B\frac{\mathrm{i}}{A-\mathrm{i}\gamma}B\right)\bar{\Pi}_0=0 \tag{8.7}$$

which is the explicit form of (4.3) for the two-level process. The solution for $\bar{\Pi}_1$ could be found from the general expression (7.18), but from (8.4) we obtain directly

$$\bar{\Pi}_1=\mathrm{i}a^2[\mathrm{i}/(A-\mathrm{i}\gamma)]B\bar{\Pi}_0. \tag{8.8}$$

Equation (8.3) relates $\bar{\Pi}_0$ and $\bar{\Pi}_1$ according to

$$B\bar{\Pi}_1=-A\bar{\Pi}_0 \tag{8.9}$$

which is consistent with (8.7) and (8.8). These steady-state solutions for the moments can again serve as the initial correlations for the correlation functions, just as in § 5.

It was pointed out in § 6 that the random-jump process reduces to Gaussian white noise if the variance and γ become infinite, but $\lambda=\langle x^2\rangle/\gamma$ remains finite. We can take this limit easily in the explicit solution (8.5) for the random telegraph. This gives

$$\tilde{\Pi}_0(\omega)=\frac{\mathrm{i}}{\omega-A+\mathrm{i}\lambda B^2}\Pi_0(t_0) \tag{8.10}$$

which is the familiar result (Fox 1972). We notice that the initial correlation $\Pi_1(t_0)$ disappears indeed in this limit.

We solved the random telegraph problem here as an example of the general theory of initial correlations. In fact the solution (8.5), including the initial correlations, can be found more directly for this specific process. To this end we first notice that the relation (2.9) for the moments reduces to

$$\langle x(t_n)\ldots x(t_1)\rangle=a^n\exp[-\gamma(t_n-t_{n-1}+\ldots+t_4-t_3+t_2-t_1)] \qquad n=2,4,6,\ldots \tag{8.11}$$

for $t_n\geqslant\ldots\geqslant t_1$, as a consequence of $x(t)^2=a^2$. The moments for $n=1,3,5,\ldots$, are zero. Equation (8.11) implies the relations

$$\begin{aligned}\langle x(t+\tau)\sigma(t)\rangle&=\mathrm{e}^{-\gamma\tau}\langle x(t)\sigma(t)\rangle\\ \langle x(t+\tau)x(t)\sigma(t)\rangle&=a^2\,\mathrm{e}^{-\gamma\tau}\langle\sigma(t)\rangle\end{aligned} \tag{8.12}$$

for $\tau\geqslant 0$, if $\sigma(t)$ is a solution of

$$\mathrm{i}\,\mathrm{d}\sigma/\mathrm{d}t=(A+x(t)B)\sigma. \tag{8.13}$$

This differential equation is identical to

$$\begin{aligned}\mathrm{i}(\mathrm{d}/\mathrm{d}t)\sigma(t)&=A\sigma(t)+B\exp[-\mathrm{i}A(t-t_0)]x(t)\sigma(t_0)\\ &\quad-\mathrm{i}B\int_{t_0}^{t}\exp[-\mathrm{i}A(t-t')]Bx(t)x(t')\sigma(t')\,\mathrm{d}t'.\end{aligned} \tag{8.14}$$

If we take the average term by term, apply (8.12) and take the Laplace transform, we obtain

$$(\omega-A)\langle\tilde{\sigma}(\omega)\rangle=\mathrm{i}\langle\sigma(t_0)\rangle+B\frac{\mathrm{i}}{\omega-A+\mathrm{i}\gamma}\langle x(t_0)\sigma(t_0)\rangle-\mathrm{i}a^2B\frac{\mathrm{i}}{\omega-A+\mathrm{i}\gamma}B\langle\tilde{\sigma}(\omega)\rangle \tag{8.15}$$

which is identical to the solution (8.5). It is remarkable that this solution has not been found before.

9. The three-state process

The method of the previous section is easily extended to the situation where $x(t)$ has the three possible values $-a$, 0, a. The probabilities are chosen as $P(-a)=P(a)$, $P(0)=1-2P(a)$ and the variance of $x(t)$ becomes

$$\langle x^2\rangle = 2a^2P(a). \tag{9.1}$$

If we choose $P(0)=0$, this three-level process reduces to the random telegraph signal from § 8. We now have the obvious relation

$$x(t)^3 = a^2x(t) \tag{9.2}$$

for every t, which implies for the third moment

$$\Pi_3(t) = a^2\Pi_1(t). \tag{9.3}$$

If we then apply (9.3) in the set of recurrence relations (7.5), we find again that the set truncates. We obtain the closed set of three coupled differential equations

$$\begin{aligned} &\mathrm{i}(\mathrm{d}/\mathrm{d}t)\Pi_0(t) = A\Pi_0(t) + B\Pi_1(t) \\ &\mathrm{i}(\mathrm{d}/\mathrm{d}t)\Pi_1(t) = (A-\mathrm{i}\gamma)\Pi_1(t) + B\Pi_2(t) \\ &\mathrm{i}(\mathrm{d}/\mathrm{d}t)\Pi_2(t) = (A-\mathrm{i}\gamma)\Pi_2(t) + a^2B\Pi_1(t) + \mathrm{i}\gamma\langle x^2\rangle\Pi_0(t) \end{aligned} \tag{9.4}$$

for the moments $\Pi_0(t)$, $\Pi_1(t)$ and $\Pi_2(t)$. It is straightforward to solve this set with a Laplace transform, just as in the previous section. The general solution contains the three arbitrary initial correlations $\Pi_0(t_0)$, $\Pi_1(t_0)$ and $\Pi_2(t_0)$, which determine the moments for $t \geq t_0$.

10. Conclusions

The multiplicative process $\sigma(t)$ is assumed to obey the stochastic differential equation $\mathrm{i}\,\mathrm{d}\sigma/\mathrm{d}t = L(x(t))\sigma$ with $x(t)$ the random-jump process. We apply the Burshtein equation for the marginal averages to solve this equation for the stochastic average $\langle\sigma(t)\rangle$. We take into account the possibility that the initial value $\sigma(t_0)$ is stochastic. The finite correlation time of the process $x(t)$ then implies that $\langle\sigma(t)\rangle$ is not only determined by the average initial state $\langle\sigma(t_0)\rangle$, but that an initial correlation operator for $\sigma(t_0)$ is required. In the case that $L(x(t))$ is linear in $x(t)$, it is sufficient to prescribe the initial moments $\Pi_k(t_0)$, as is demonstrated explicitly for the bivalued process, the random telegraph. Then the solution is determined by $\Pi_0(t_0)$ and $\Pi_1(t_0)$, and it is pointed out how this can be generalised to the three-state jump process.

The steady-state solution $\langle\sigma(t=\infty)\rangle$ is independent of the initial correlations, and is fully determined by $L(x(t))$ and the stochastics of $x(t)$. We derived the equation for this steady-state value. Subsequently it was shown that the stationary (quantum) correlation function of two observables can be expressed as a single-time average times this steady state. Due to the finite correlation time of the $x(t)$ process, these correlation functions do not factorise in the average of the regression operator times the average equal-time correlation, as is commonly assumed as an approximation.

Appendix

In this paper we presented an exact and operational theory of multiplicative stochastic processes with random jumps, which can be applied directly in practical cases. In this appendix we discuss in more detail the effects of the finite correlation time γ^{-1} and the structure of the solution.

The general result (3.8) for $\langle\tilde{\sigma}(\omega)\rangle$ can be transformed to the time domain, which yields the integral equation (4.1). If we differentiate (4.1) with respect to t, we obtain

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\langle\sigma(t)\rangle = \gamma\int_{t_0}^{t}\mathrm{d}t'\exp[-\gamma(t-t')]\int\mathrm{d}x\,P(x)L(x)\exp[-\mathrm{i}L(x)(t-t')]\langle\sigma(t')\rangle$$
$$+\exp[-\gamma(t-t_0)]\langle L(x(t_0))\exp[-\mathrm{i}L(x(t_0))(t-t_0)]\sigma(t_0)\rangle \tag{A1}$$

and a Laplace transform of this equation gives

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{\mathrm{i}}{\omega - L_{\mathrm{av}}(\omega)}\langle\sigma(t_0)\rangle + \frac{1}{\omega - L_{\mathrm{av}}(\omega)}\left\langle L(x(t_0))\frac{\mathrm{i}}{\omega - L(x(t_0)) + \mathrm{i}\gamma}\sigma(t_0)\right\rangle \tag{A2}$$

where we introduced the operator

$$L_{\mathrm{av}}(\omega) = \int\mathrm{d}x\,P(x)L(x)\frac{\mathrm{i}\gamma}{\omega - L(x) + \mathrm{i}\gamma}. \tag{A3}$$

The result (A2) is equivalent to (3.8). We notice however that (A2) involves a different initial correlation operator, in comparison with (3.8), and that the average initial state $\langle\sigma(t_0)\rangle$ enters the solution.

Let us now consider the limit of a small correlation time γ^{-1}. We obviously have

$$\lim_{\gamma^{-1}\to 0} L_{\mathrm{av}}(\omega) = \int\mathrm{d}x\,P(x)L(x) = \bar{L} \tag{A4}$$

$$\lim_{\gamma^{-1}\to 0}\left\langle L(x(t_0))\frac{\mathrm{i}}{\omega - L(x(t_0)) + \mathrm{i}\gamma}\sigma(t_0)\right\rangle = 0 \tag{A5}$$

and (A2) reduces to

$$\langle\tilde{\sigma}(\omega)\rangle = \frac{\mathrm{i}}{\omega - \bar{L}}\langle\sigma(t_0)\rangle. \tag{A6}$$

The Laplace inverse is the sample exponential

$$\langle\sigma(t)\rangle = \exp[-\mathrm{i}(t-t_0)\bar{L}]\langle\sigma(t_0)\rangle. \tag{A7}$$

The first effect of the finite correlation time is that the $\bar{L}$ in (A6) turns into the frequency-dependent operator $L_{\mathrm{av}}(\omega)$. This introduces a memory in the time evolution for the average, but $\langle\sigma(t)\rangle$ is still determined by $\langle\sigma(t_0)\rangle$ only. The second effect of $\gamma^{-1}\neq 0$ is the appearance of the initial correlation operator in (A2). This quantity involves the stochastic $\sigma(t_0)$, which also depends on the history of the process, and hence knowledge of $\langle\sigma(t_0)\rangle$ only is not sufficient any more. These two distinct features are clearly separated in (A2). We notice that the structure of (A2) is very similar to the expression for a density operator of an atom in a perturber bath (Nienhuis 1982). There the $L_{\mathrm{av}}(\omega)$ is the binary-collision operator. Its frequency dependence and the appearance of the initial correlations are then a consequence of the finite collision time. In the more familiar impact limit of a collision, the collision time approaches zero and $L_{av}(\omega)$ reduces to an effective ω-independent collision operator and the initial correlations vanish.

From (A2) we can also derive an equation for the steady state $\bar{\sigma}$. We find simply

$$L_{av}(0)\bar{\sigma} = 0 \tag{A8}$$

which is identical to (4.3) because of the identity

$$L_{av}(0) = -i\gamma(1 - \gamma G(i\gamma)). \tag{A9}$$

Equation (A8) reveals clearly that the average steady state $\bar{\sigma}$ is determined by the operator $L_{av}(\omega)$, which accounts for the evolution of the average state $\langle\sigma(t)\rangle$. This operator $L_{av}(0)$ is however not equal to the average of $L(x(t))$, which is $\bar{L}$.

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